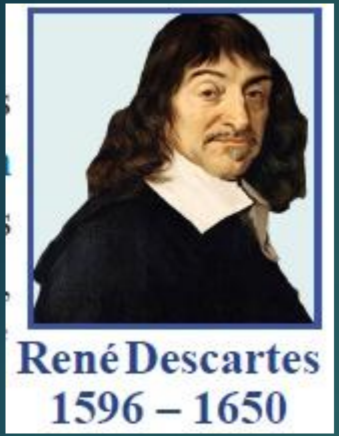
"Divide each difficulty into as many parts as is feasible and necessary to resolve it"

René Descartes



TWO DIMENSIONAL ANALYTICAL GEOMETRY - II

Introduction

- Analytical geometry of two dimension is used to describe geometric objects such as,
  - Point
  - Line
  - Circle
  - Parabola
  - Ellipse
  - Hyperbola

  These are all Cartesian coordinate system.

  The 2000 year ago the ancient Greeks studied conic curves.

Analytical geometry was systematically developed by

- Kepler
- Newton
- Euler
- leibniz
- L ' Hospital
- Clairaut
- Cramer
- Jacobi's

- Analytical geometry grew out of need for establishing algebraic techniques for solving geometric problem.
- The development in this area as conquered industry medicine and scientific research.
- The theory of planetary motions developed by Johannes Kepler.
- The German mathematician cum physicist starting that all the planets in the solar system including the earth are moving in elliptical orbit with sun one of the focus.

Circle

Definition:

a circle is the locus of a point in a plane which moves such that its distance from a fixed point in the plane is always a constant.

a fixed point is called the center and the constant distance is called radius of the circle.

Equation of the circle of the standard form

- Equation of circle with center (0,0) and radius r.$X^2+Y^2 = r^2$.

- Equation of the circle with center h,k and radius r. $(x-h)^2+(y-k)^2 = r^2$.

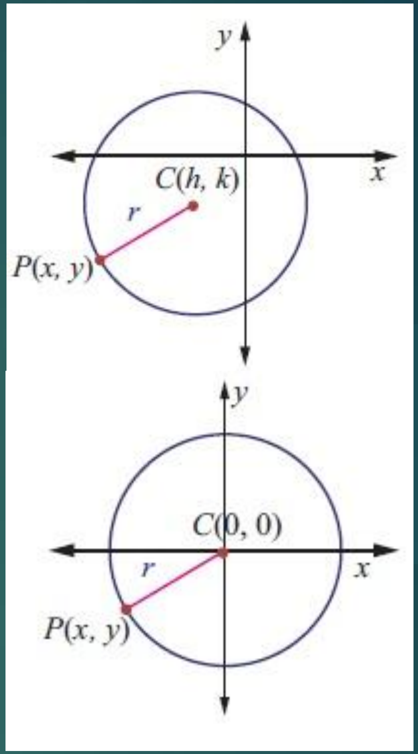
- The equation $X^2+Y^2 + 2gx+ 2fy+ c = 0$ is a second degree

in x and y possessing the following characteristics.

i. it is a second degree equation in x and y.

ii. Coefficient of x^2 = Coefficient of $y^2 \neq 0$.

iii. Coefficient of x y = 0.



Remark

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents

a real circle if $g^2 + f^2 - c > 0$;

a point circle if $g^2 + f^2 - c = 0$;

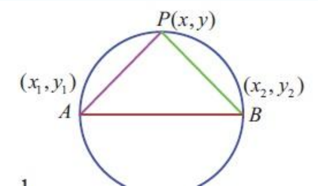
an imaginary circle if $g^2 + f^2 - c < 0$ with no locus.

Theorem 5.1

The circle passing through the points of intersection of the line $lx + my + n = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is the circle of the form $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$, $\lambda \in \mathbb{R}^1$.

Theorem 5.2

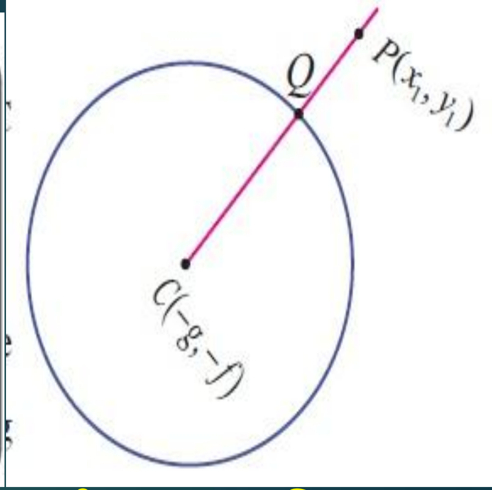
The equation of a circle with (x_1, y_1) and (x_2, y_2) as extremities of one of the diameters of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.



Theorem 5.3

The position of a point $P(x_1, y_1)$ with respect to a given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in the plane containing the circle is outside or on or inside the circle according as

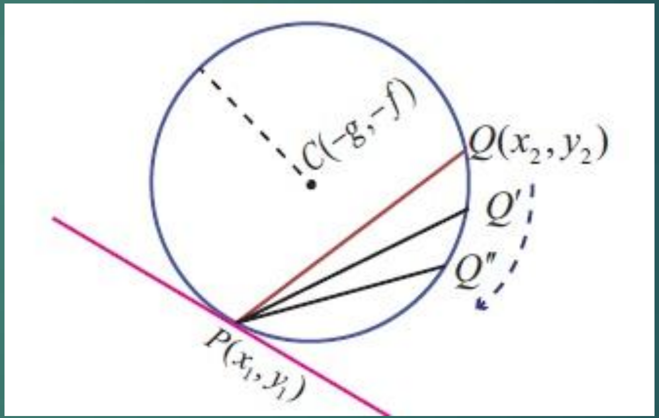
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \text{ is } \begin{cases} > 0 & \text{or,} \\ = 0 & \text{or,} \\ < 0. & \end{cases}$$



Equation of tangents and normal at a point of circle

- Equation of the circle

$X^2 + Y^2 + 2gx + 2fy + c = 0$



Equation of the tangents

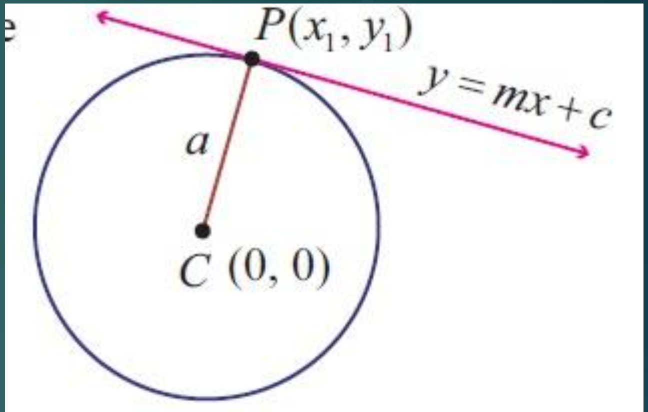
$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0.$

- Equation of the normal

$yx_1 - xy_1 + g(y-y_1) - f(x-x_1) + c = 0$

Condition for the line y=mx+c to be a tangent to the circle $x^2+y^2 = a^2$and finding the point of contact

- Let the line y=mx+c touch the circle $x^2+y^2= a^2$.
- The centre and radius of the circle $x^2+y^2= a^2$ are (0,0) and a respectively.

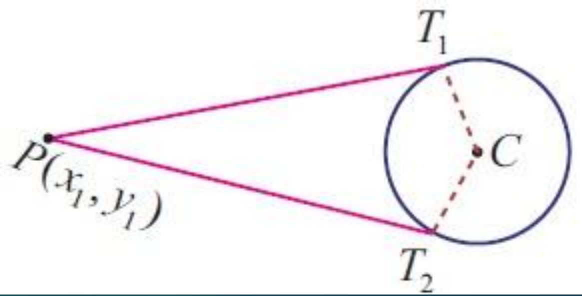


(i) condition for line to be tangent

$$\frac{|c|}{\sqrt{1+m^2}} = a \text{ (or) } c^2 = a^2(1+n^2)$$

Theorem :5.4

From any point outside the circle $x^2+y^2= a^2$two tangents can be drawn.



Conic

Definition :

- A conic is the locus of the point which moves in a plane show that its distance from a fixed points bears a constants ratio to its distance from a fixed line not containing a fixed point.
- The fixed point is called in focus.
- The fixed line is called in directrix
- The constants ratio is called eccentricity, which is denoted by e

i. If this constant $e = 1$ then the conic is called a parabola.

ii. If this constant $e < 1$ then the conic is called a ellipse.

iii. If this constant $e > 1$ then the conic is called a hyperbola.

The general equation of a conic

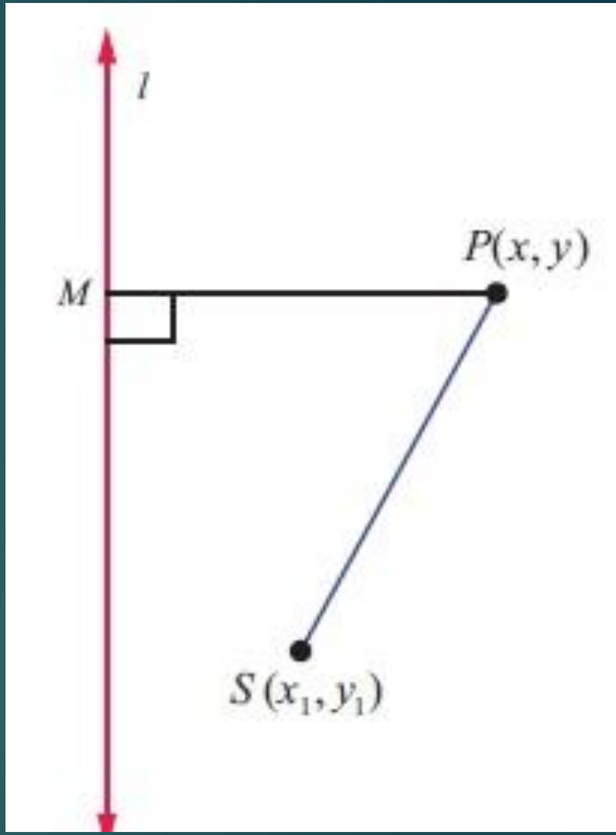
- $(x-x_1)^2+(y-y_1)^2 = e^2\left[\frac{lx+my+n}{\sqrt{l^2+m^2}}\right]^2$

- Yielding the following the cases:

(i)$B^2 - 4AC = 0 \Leftrightarrow e = 1 \Leftrightarrow$ the conic is Parabola.

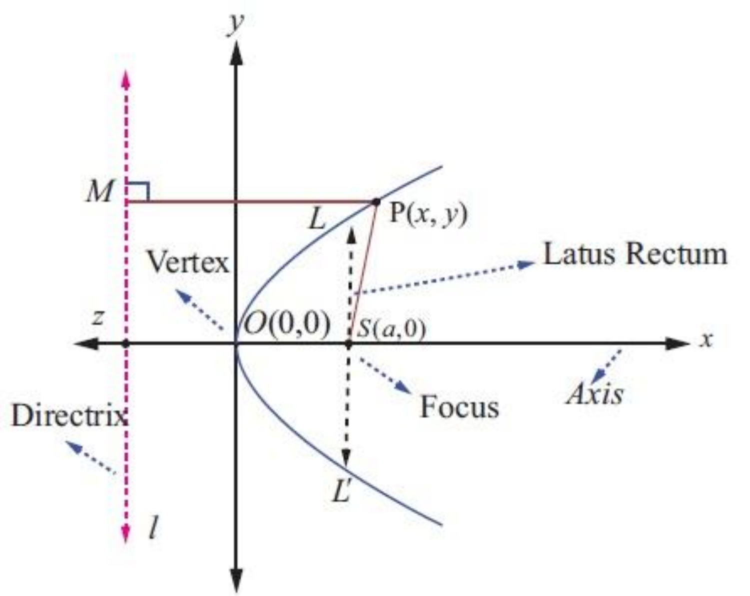
(ii) $B^2 - 4AC < 0 \Leftrightarrow 0 < e < 1 \Leftrightarrow$ the conic is Ellipse.

(iii) $B^2 - 4AC > 0 \Leftrightarrow e > 1 \Leftrightarrow$ the conic is Hyperbola.



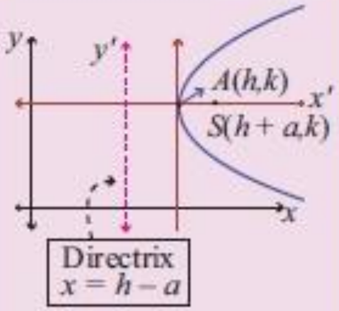
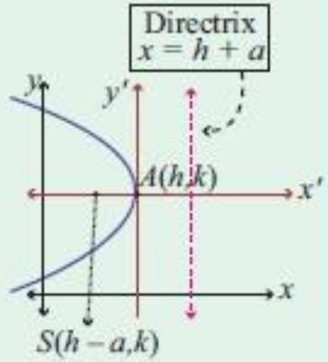
Parabola

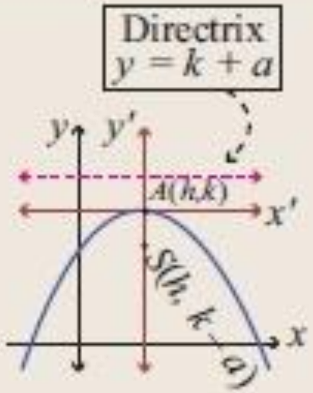
- Equation of the parabola in standard form with vertex at (0,0)

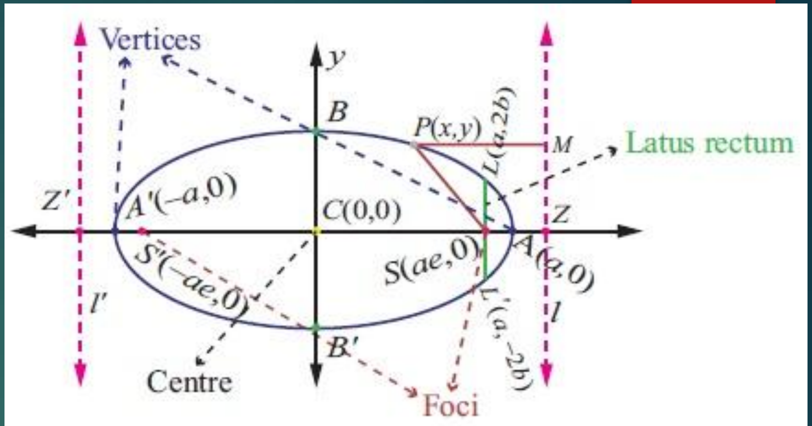


Definition 5.3

- The line perpendicular to the directrix and passing through the focus is known as the **Axis** of the parabola.
- The intersection point of the axis with the curve is called **vertex** of the parabola
- Any chord of the parabola, through its focus is called **focal chord** of the parabola
- The length of the focal chord perpendicular to the axis is called **latus rectum** of the parabola

Equation	Graph	Vertices	Focus	Axis of symmetry	Equation of directrix	Length of latus rectum
$(y-k)^2 = 4a(x-h)$	 (a) The graph of $(y-k)^2 = 4a(x-h)$ Fig. 5.19	(h,k)	$(h+a,0+k)$	$y=k$	$x=h-a$	$4a$
$(y-k)^2 = -4a(x-h)$	 (b) The graph of $(y-k)^2 = -4a(x-h)$ Fig. 5.20	(h,k)	$(h-a,0+k)$	$y=k$	$x=h+a$	$4a$

$(x-h)^2 = 4a(y-k)$	 (c) The graph of $(x-h)^2 = 4a(y-k)$ Fig. 5.21	(h,k)	$(0+h, a+k)$	$x = h$	$y = k - a$	$4a$
$(x-h)^2 = -4a(y-k)$	 (d) The graph of $(x-h)^2 = -4a(y-k)$ Fig. 5.22	(h,k)	$(0+h, -a+k)$	$x = h$	$y = k + a$	$4a$



Ellipse

- An ellipse is the locus of a point which moves such that it is distance from a fixed point (focus) bears a constant ratio (eccentricity)less than unity its distance from its directrix bearing a constant ratio $0 < e < 1$.
- equation of an ellipse in standard form

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which the equation of an Ellipse in standard form it is symmetrical about x and y axis.

Definition 5.4

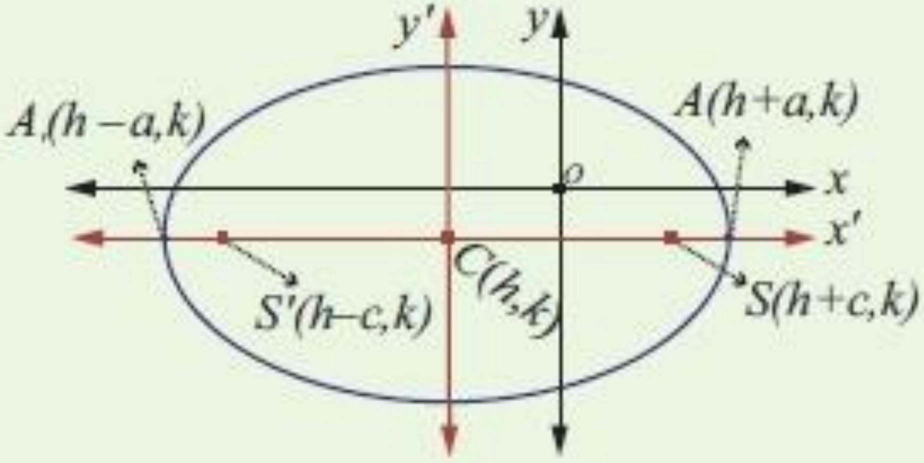
(1) The line segment AA' is called the **major axis** of the ellipse and is of length $2a$.

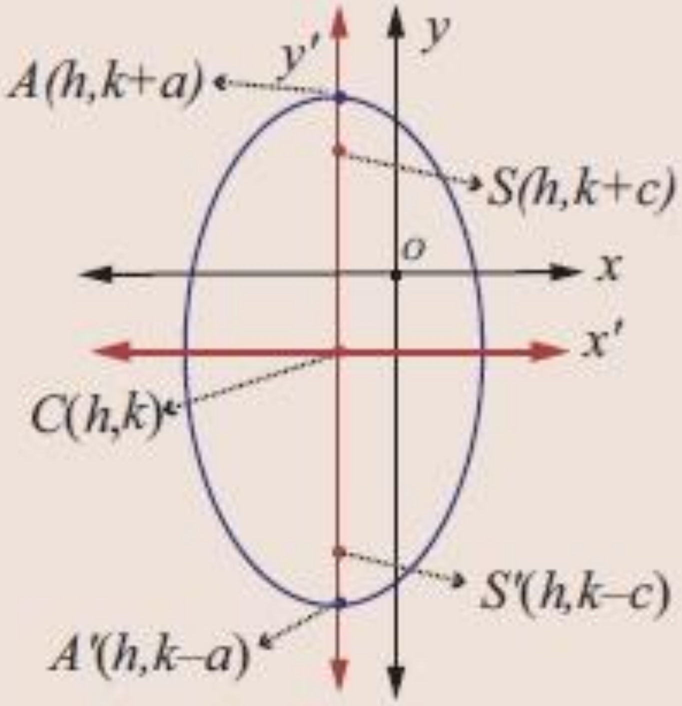
(2) The line segment BB' is called the **minor axis** of the ellipse and is of length $2b$.

(3) The line segment $CA =$ the line segment $CA' =$ **semi major axis** $= a$ and the line segment $CB =$ the line segment $CB' =$ **semi minor axis** $= b$.

(4) By symmetry, taking $S'(-ae, 0)$ as focus and $x = -\frac{a}{e}$ as directrix l' gives the same ellipse.

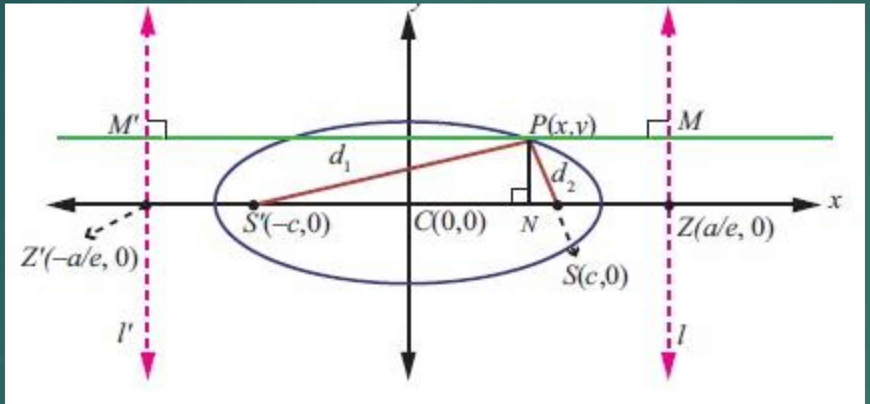
Thus, we see that an ellipse has two foci, $S(ae, 0)$ and $S'(-ae, 0)$ and two vertices $A(a, 0)$ and $A'(-a, 0)$ and also two directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Equation	Centre	Major Axis	Vertices	Foci
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \boxed{a^2>b^2}$  Fig.5.24 (a) Major axis parallel to the x-axis. Foci are c units right and c units left of centre, where $c^2=a^2-b^2$.	(h,k)	parallel to the x-axis	$(h-a,k)$ $(h+a,k)$	$(h-c,k)$ $(h+c,k)$

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $\boxed{a^2>b^2}$  Fig.5.25 (b) Major axis parallel to the y-axis. Foci are c units right and c units left of centre, where $c^2=a^2-b^2$.	(h,k)	parallel to the y-axis	$(h,k-a)$ $(h,k+a)$	$(h,k-c)$ $(h,k+c)$

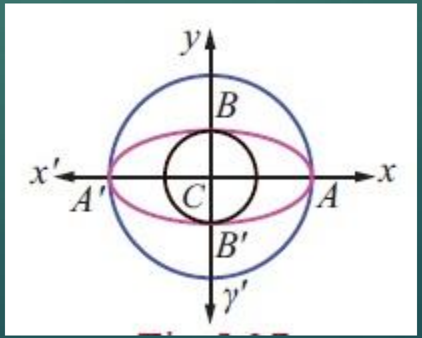
Theorem :5.5

The sum of the focus distance of any point on the Ellipse is equal to length of the major axis.



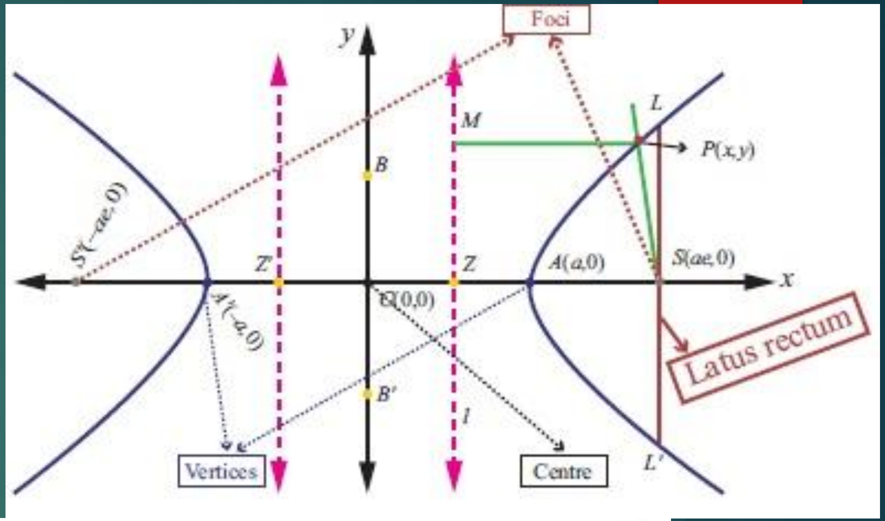
Remarks:

- Auxiliary circle or circumcircle is the circle with length of major axis as diameter and incircle is the circle with length of minor axis as diameter .
- $x^2+y^2= a^2$ and $x^2+y^2= b^2$ respectively.



Hyperbola

- Equation of a hyperbola in standard form with centre at (0,0).



Definition 5.5

(1) The line segment AA' is the **transverse** axis of length $2a$.

(2) The line segment BB' is the **conjugate** axis of length $2b$.

(3) The line segment $CA =$ the line segment $CA' =$ **semi transverse axis** $= a$ and the line segment $CB =$ the line segment $CB' =$ **semi conjugate axis** $= b$.

(4) By symmetry, taking $S'(-ae,0)$ as focus and $x = -\frac{a}{e}$ as directrix l' gives the same hyperbola.

Thus we see that a hyperbola has two foci $S(ae,0)$ and $S'(-ae,0)$, two vertices $A(a,0)$ and $A'(-a,0)$ and two directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

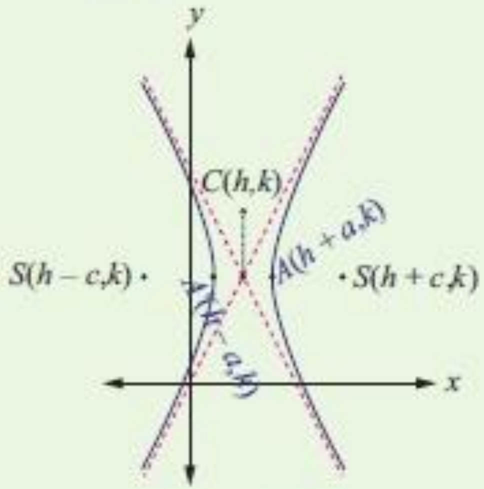


Fig. 5.29
(a) transverse axis parallel to the x-axis

(a) Transverse axis parallel to the x-axis.

The equation of a hyperbola with centre $C(h,k)$ and transverse axis parallel to the x-axis (Fig. 5.29) is given by $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

The coordinates of the vertices are $A(h+a,k)$ and $A'(h-a,k)$. The coordinates of the foci are $S(h+c,k)$ and $S'(h-c,k)$ where $c^2=a^2+b^2$.

The equations of directrices are $x=\pm\frac{a}{e}$.

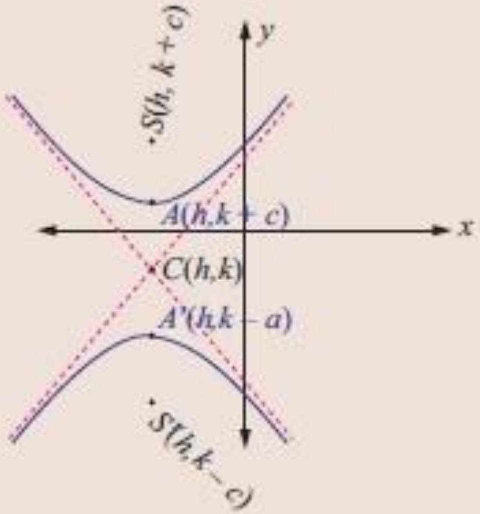


Fig. 5.30
(b) transverse axis parallel to the y-axis

(b) Transverse axis parallel to the y-axis

The equation of a hyperbola with centre $C(h,k)$ and transverse axis parallel to the y-axis (Fig. 5.0) is given by

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

The coordinates of the vertices are $A(h,k+a)$ and $A'(h,k-a)$. The coordinates of the foci are $S(h,k+c)$ and $S'(h,k-c)$, where $c^2=a^2+b^2$.

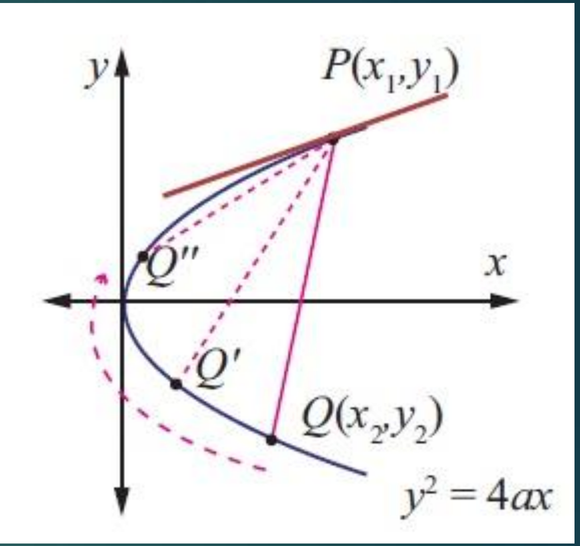
The equations of directrices are $y=\pm\frac{a}{e}$.

Parametric form of conics

-
- Parametric form of the circle $x^2+y^2=a^2$.
- Parametric form of the parabola $y^2 = 4ax$.
- Parametric form of the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- Parametric form of the Hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Tangents and normal to conics

- Equation of tangents and normal to the parabola $y^2 = 4ax$.
- Equation of tangent in parametric form $yt = x+at^2$.
- Equation of normal in Cartesian form $xy_1+2ay = x_1y_1 + 2ay_1$.
- Equation of normal in parametric form $y+xt = at^3+2at$.



Theorem: 5.6

- Three normal can be drawn to parabola $y^2=4ax$ from a given point one of which is always real.

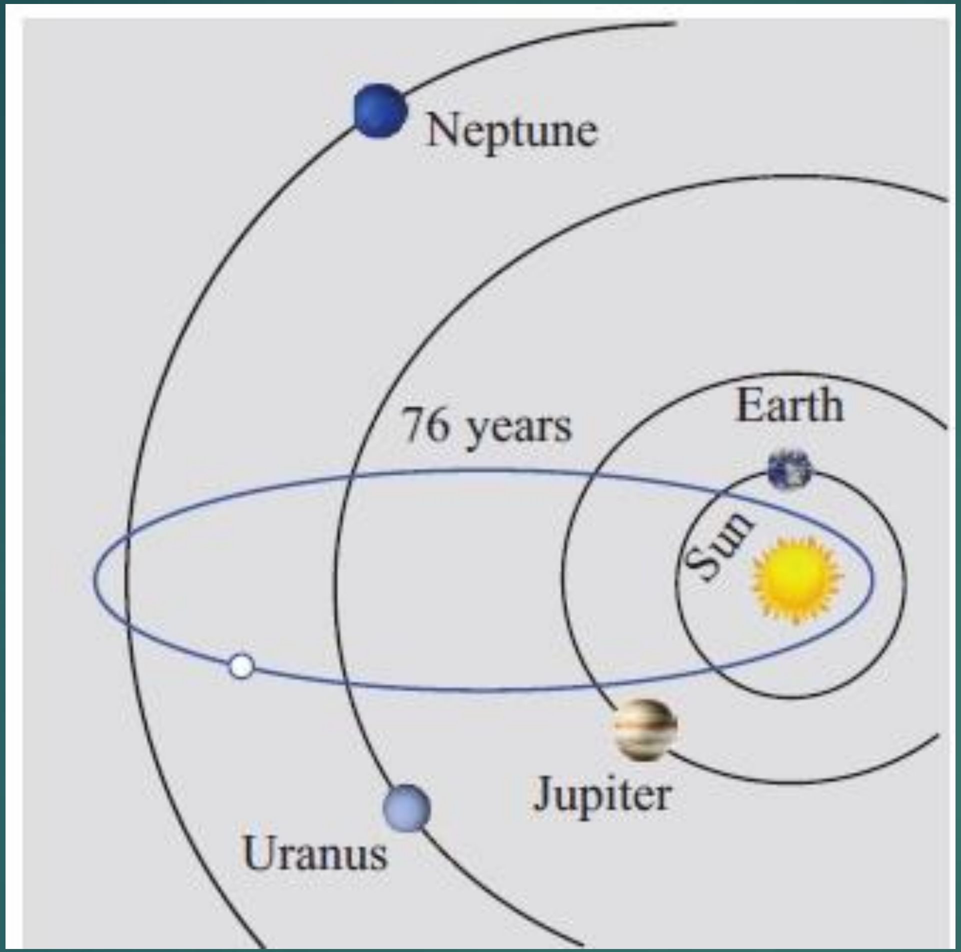
Real life of Conics

- Parabola

Bridge of river in Godavari ,

Eiffel tower

Ellipse

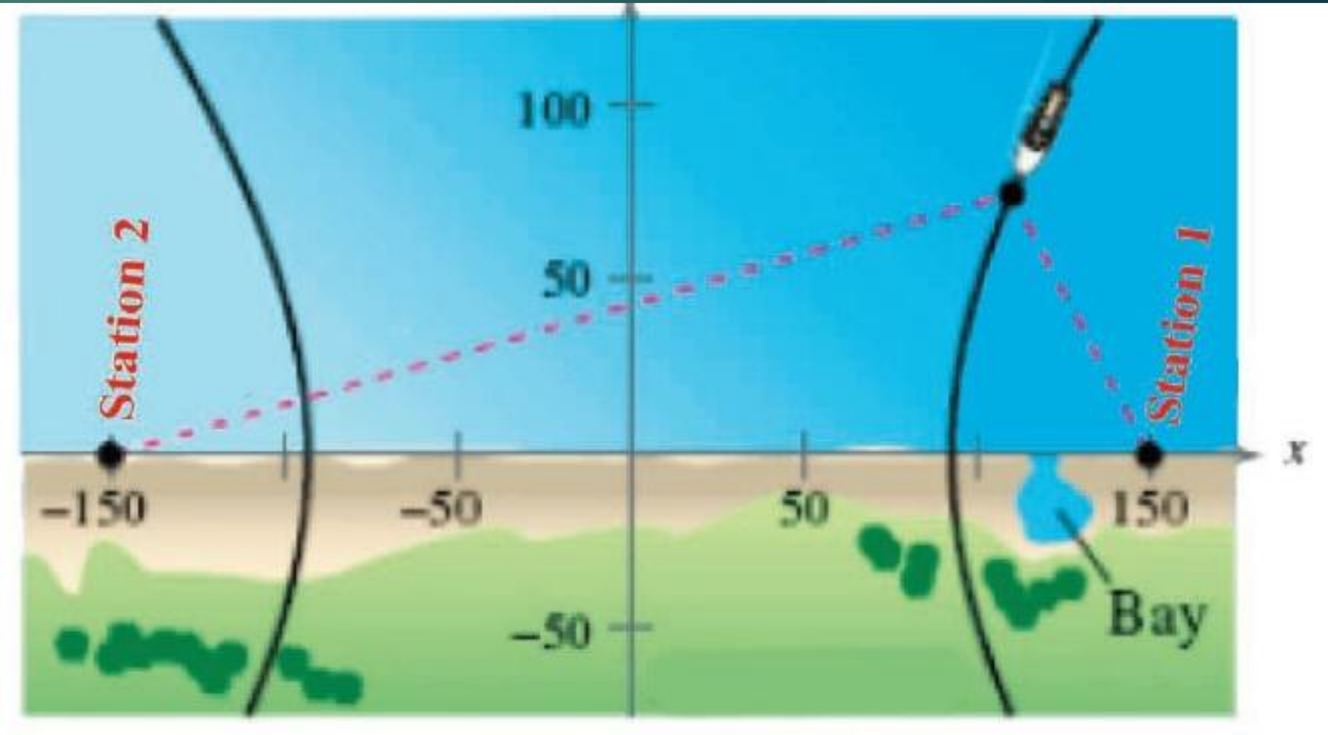


- The elliptical orbit of Halley's comet

Hyperbola

Mumbai airport terminal,

an locating ships



Thank you